Channel Capacity of Analog Network Coding in a Wireless Channel

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***Abstract*—Radio channel is characterized by a multipath propagation that is suitable for application of network coding (NC) to improve the network capacity. In particular, physical layer network coding (PNC) scheme is known to double the network capacity of bi-directional communication between pairs of end users assisted by a relay terminal in a Gaussian channel. Recently, analog network coding (ANC) has been proposed with a simpler implementation complexity in comparison with PNC scheme, but the same achievable network capacity. On the other hand, an important question is how much the channel capacity is improved over a conventional point-to-point communication in a frequency-selective fading channel. In this paper, we present the ergodic channel capacity analysis of bi-directional transmission with analog network coding in a frequency-selective fading channel. The channel capacity expression for transmission with frequency domain equalization (FDE) is derived based on the Gaussian approximation of the inter-symbol interference (ISI) after FDE.**

***Index Terms*—Analog network coding, channel capacity, multipath fading.**

I. Introduction

The next-generation communication networks are characterized by a broadband radio access with a large area caverage and a high capacity for end-users. In wired networks [1], a network coding had been studied in the context of distributed source coding as a promising technique to improve the network capacity based on a logical flow operations on the network layer. As stated before the original investigation of network coding was in the context of wired networks, however, its potential to improve the performance in wireless communication networks arise due to the broadcast nature of the wireless propagation environment [2], [3].

Network coding at the physical layer in a wireless relay network increases the network capacity of bi-directional communication [4] (bi-directional amplification of the throughput (BAT-relaying)), [5] (physical layer network coding (PNC)). The principle of PNC in combination with simple relaying over a Gaussian channel was investigated in [6]. In [7], the PNC performance is evaluated over a Gaussian channel with symmetric channels between sources and relay. In [8], PNC with design of encoding/decoding over a Gaussian channel is considered to maximize capacity, while in [9] bi-directional sum rate is investigated. PNC with relay using multiple antennas over a flat fading symmetric channels is presented, in [10], to maximize the achievable sum rate. In [11], PNC scheme where the relay estimates source signals from the superimposed signal and combines them over complex field rather than Galois field before forwarding is presented. However, the implementation of these PNC schemes is complex. The analog network coding (ANC) proposed in [12] is essentially another variation of PNC with a simpler implementation. To date, this method has been addressed in the literature mainly within information theory group for bi-directional communication over a Gaussian symmetric (i.e., link between users and relay and vice verse is assumed identical) channel. The channel capacity over a wireless multipath (i.e., frequency-selective) channel has not been investigated.

Most of the work on ANC (and/or PNC) points out its advantage that the network capacity is doubled in comparison with conventional point-to-point communication. However, more important question, from the wireless communication perspective, is how much the channel capacity can be improved over a conventional point-to-point and cooperative relay communication. To the best of the authors knowledge such evaluation has not been presented.

In this paper, we evaluate the channel capacity of ANC in a wireless channel. The wireless channel is characterized as a multipath propagation environment (i.e., a frequency-selective fading) giving rise to inter-symbol interference (ISI) that may significantly degrade the system performance [13]. In such environment two schemes may be considered: (i) orthogonal frequency division multiplexing (OFDM) [14] and (ii) single carrier with frequency domain equalization (SC-FDE) [15]. We evaluate the ergoding channel capacity, defined as the ensamble average of the information rate over the channel distribution, of both schemes in a frequency selective fading channel by numerical analysis and compare their achievable performances with a point-to-point communication and cooperative relaying. The importance of the ergodic capacity is that for every independent channel use, a transmission with the rate defined by ergodic capacity is possible with the error close to zero assuming that asymptotically optimal codebooks are used. In the case of SC-FDE, the expression for ergodic channel capacity is given based on the Gaussian approximation of the inter-symbol interference (ISI) after FDE. On the other hand, in OFDM, the ISI is removed by the use of cyclic prefix within the guard interval.

The remainder of this paper is organized as follows. Section

II gives an overview of network model. The capacity analysis is presented in Sect. III. In Sect. IV, numerical results and discussions are presented. Section V concludes the paper.

II. Network Model

We assume that coverage area of terminals T_0 and T_1 include an access point (henceforth relay terminal), while they are out of each other's transmission range. R denotes the relay terminal using amplify-and-forward network protocol [16] that is used in relay cooperative networks [17]. Each terminal is equipped with an omni-directional antenna, and the channel is half duplex so that transmission and reception between sources and relay must occur in different time slots.

We assume that communication takes place in two stages as shown in Fig. 1. During the first stage, terminals T_0 and T_1 transmit in the same time to the relay. During the second stage, the relay broadcast to both terminals T_0 and T_1. It is assumed that communication in the first and second stages are orthogonal. This could be done in two separate time slots and hence, the first and second stage does not interfere. Without loss of generality, this can be achieved using time division multiplexing (TDM). The protocol is summarized in Table I.

The jth (j=0, 1) terminal (i.e., T_j) data-modulated symbol sequence $\{d_j(n);\ n = 0 \sim N_c - 1\}$ is chosen from a complex-valued finite constellation such as quadrature phase shift keying (QPSK) modulation. The transmit signal can be expressed using the equivalent low-pass representation as

$$s_j(t) = \begin{cases} \sqrt{P_s} d_j(t), & \text{SC} \\ \sqrt{\frac{P_s}{N_c}} \sum_{n=0}^{N_c-1} d_j(n) \exp\left\{ j2\pi t \frac{n}{N_c} \right\}, & \text{OFDM} \end{cases} \tag{1}$$

for $t = 0 \sim N_c - 1$, where P_s $(= E_s/T_s)$ denotes the source transmit power. E_s and T_c denote the data-modulated symbol energy and the symbol duration, respectively. After insertion of N_g-sample guard interval (GI) the signal is transmitted over a frequency-selective fading channel.

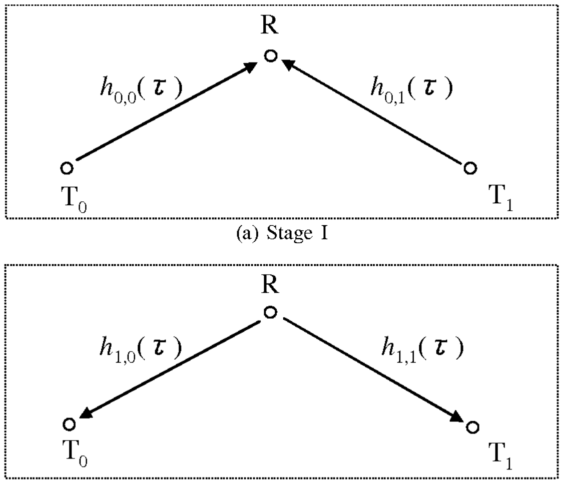
The signal propagates through the channel with a discrete-time channel impulse response $h_{m,j}(\tau)$ given by

$$h_{m,j}(\tau) = \sum_{l=0}^{L-1} h_{l,m,j} \delta(\tau - \tau_l), \tag{2}$$

where L, $h_{l,m,j}$, τ_l and $\delta(t)$ denote the number of paths, the path gain between the jth terminal T_j and relay at stage m (m=0, 1), the time delay of the lth path and the delta function. Without loss of generality, we assume $\tau_0 = 0 < \tau_1 < \cdots < \tau_{L-1}$ and that the lth path time delay is $\tau_l = l\Delta$, where Δ (≥ 1) denotes the time delay separation between adjacent paths.

Stage I: The signal transmitted by the source terminals during the first stage is given by Eq. (1). The received signal at the relay terminal during the first stage may be represented in the frequency domain as

$$R_r(n) = \begin{cases} S_0(n)H_{0,0}(n) + S_1(n)H_{0,1}(n) + N_r(n), & \text{SC} \\ d_0(n)H_{0,0}(n) + d_1(n)H_{0,1}(n) + N_r(n), & \text{OFDM} \end{cases} \tag{3}$$



(a) Stage I

(b) Stage II

Fig. 1. Physical layer network coding.

for the nth subcarrier (frequency, in the case of SC-FDE) ($n = 0 \sim N_c - 1$), where $S_j(n)$, $H_{0,j}(n)$ and $N_r(n)$ denote the Fourier transforms of the jth terminal transmit signal, the channel gain between T_j and R and the additive white Gaussian noise (AWGN) having power spectral density N_0, respectively.

The relay terminal amplifies the received signal (i.e., Eq. (3) by a factor of $\sqrt{P_s}$ and retransmits the signal $\tilde{R}_r(n)$ in the time domain during the second time slot. This is equivalent as if N_c-point IFFT is applied to $\tilde{R}_r(n)$ before transmission. We assume that the sources and the relay transmit with the same power (the source transmits with half of the total available power.

Stage II: At the jth terminal T_j, an N_c-point FFT is applied to decompose the received signal into N_c subcarrier (frequency, in the case of SC-FDE) components represented by $\{R_j(n);\ n = 0 \sim N_c - 1\}$, where $j \in \{0, 1\}$ (we define $\bar{j} \in \{0, 1\}$, where the bar over the expression signifies the unitary complement operation (i.e., "NOT" operation) that performs logical negation of the value under the bar). The jth terminal T_j receives the relayed signal during the second stage represented as

$$R_j(n) = \tilde{R}_r(n) H_{1,j}(n) + N_j(n). \tag{4}$$

The jth terminal T_j removes its self information from the received signal as

$$\tilde{R}_j(n) = \begin{cases} R_j(n) - S_j(n)H_{0,j}(n)H_{1,j}(n), & \text{SC} \\ R_j(n) - d_j(n)H_{0,j}(n)H_{1,j}(n), & \text{OFDM.} \end{cases} \tag{5}$$

One-tap equalization is done in the frequency domain to combat the channel impairments. The equalized signal can be represented as

$$\hat{R}_j(n) = \tilde{R}_j(n) w_j(n) = \begin{cases} S_{\bar{j}}(n)\hat{H}_{0,\bar{j}}(n)\hat{H}_{1,j}(n) + \hat{H}_{1,j}(n)\hat{N}_r(n) + \hat{N}_j(n), & \text{SC} \\ d_{\bar{j}}(n)\hat{H}_{0,\bar{j}}(n)\hat{H}_{1,j}(n) + \hat{H}_{1,j}(n)\hat{N}_r(n) + \hat{N}_j(n), & \text{OFDM,} \end{cases} \tag{6}$$

TABLE I
NETWORK PROTOCOL.

First stage	Second stage
$S_1,\ S_2\ \rightarrow\ R$	$R\ \rightarrow\ S_1,\ S_2$

where

$$\begin{cases}\hat{H}_{0,\bar{j}}(n)=H_{0,\bar{j}}(n)w_{0,\bar{j}}(n),\\ \hat{H}_{1,j}(n)=H_{1,j}(n)w_{1,j}(n),\\ \hat{N}_j(n)=N_j(n)w_{0,\bar{j}}(n)w_{1,j}(n),\\ \hat{N}_r(n)=N_r(n)w_{0,\bar{j}}(n).\end{cases} \tag{7}$$

In Eq. (6), $w_j(n)$ represents the equalization weight at the jth terminal T_j given by

$$w_j(n)=\begin{cases}\frac{H^*_{0,\bar{j}}(n)H^*_{1,j}(n)}{|H_{0,\bar{j}}(n)H_{1,j}(n)|^2+(|H_{0,\bar{j}}|^2+|H_{1,j}|^2+1)(\frac{E_s}{N_0})^{-1}}, & \text{SC}\\ \frac{H^*_{0,\bar{j}}(n)H^*_{1,j}(n)}{|H_{0,\bar{j}}(n)H_{1,j}(n)|^2}, & \text{OFDM}\end{cases} \tag{8}$$

for $n=0\sim N_c-1$. We note here that, in the above expression, MMSE-FDE weight is used for SC-FDE transmission, while in the case of OFDM we apply one-tap zero forcing (ZF) FDE.

In the case of SC-FDE transmission N_c-point IFFT is applied to Eq. (6) to obtain decision variables for data detection [15] as

$$\hat{d}_j(t)=\begin{cases}\frac{1}{\sqrt{N_c}}\sum_{n=0}^{N_c-1}\hat{R}_j(n)\exp\left\{j2\pi t\frac{n}{N_c}\right\}, & \text{SC-FDE}\\ \hat{R}_j(t), & \text{OFDM}\end{cases} \tag{9}$$

for $t=0\sim N_c-1$.

III. Ergodic Channel Capacity

In this section, we derive the expressions for conditional signal-to-interference plus noise power ratio (SINR), first, and then, the ergodic channel capacity expressions are presented. We assume that relay terminal normalizes the received signal (Eq. (3)) by a factor of $\sqrt{E[|R_r(n)|^2]}$ so that the average energy is unity, while the destination normalizes the received signal by a factor of $\sqrt{E[|R_j(n)|^2]}$ (the normalization will not alter the SINR, but it will assist theoretical derivations). We assume ideal knowledge of the channel state information with perfect self-information cancelation.

The decision variables at the jth terminal T_j after equalization can be expressed using Eq. (9) and (6) as

$$\hat{d}_j(n)=\begin{cases}d_{\bar{j}}(n)\tilde{H}+I(n)+\sum_{n=0}^{N_c-1}[\hat{H}_{1,j}(n)\hat{N}_r(n)+\hat{N}_j(n)], & \text{SC}\\ \hat{R}_j(n), & \text{OFDM},\end{cases} \tag{10}$$

where

$$\begin{cases}\tilde{H}=\frac{1}{N_c}\sum_{n=0}^{N_c-1}\hat{H}_{0,\bar{j}}(n)\hat{H}_{1,j}(n)\\ I(n)=\frac{1}{N_c}\sum_{n=0}^{N_c-1}S(n)\left[\hat{H}_{0,\bar{j}}(n)\hat{H}_{1,j}(n)-\tilde{H}\right].\end{cases} \tag{11}$$

In Eq. (10), $I(n)$ denotes the residual ISI after MMSE-FDE (note that ISI is not present in OFDM case due to insertion of cyclic prefix within the GI).

In the case of SC-FDE we assume that the residual ISI after FDE can be approximated as a zero-mean complex-valued Gaussian variable; the sum of the residual ISI and noise due to the AWGN can be treated as a new zero-mean complex-valued Gaussian noise with variance

$$2\sigma^2=2\sigma^2_{isi}+2\sigma^2_n. \tag{12}$$

Using Eqs. (10) and (13) the conditional SINR $\gamma_j(E_s/N_0,\{H_{m,j}(n)\})$ at the jth terminal T_j is represented by Eq. (14), where E_s/N_0 denotes the average signal energy per symbol-to-AWGN power spectrum density ratio.

The ergodic capacity is the ensamble average of the information rate over the channel distribution. The significance of the ergodic capacity is that for every independent (realization) channel use, we can transmit at the rate defined by ergodic capacity with error close to zero assuming that asymptotically optimal codebooks are used.

When the channel is not known at the transmitter, for the given the average E_s/N_0, the jth terminal's (T_j's) ergodic channel capacity $C_j(E_s/N_0)$ in bps/Hz can be computed as [13]

$$\begin{aligned}C_j[E_s/N_0] &= E\left[C_j\left(\frac{E_s}{N_0},\{H_{m,j}(n)\}\right)\right]\\ &= \int_0^{\infty}\cdots\int_0^{\infty}C_j\left(\frac{E_s}{N_0},\{H_{m,j}(n)\}\right)\\ &\times\ \wp[\{H_{m,j}(n)\}]\prod_n dH_{m,j}(n),\end{aligned} \tag{15}$$

where $C_j(E_s/N_0,\{H_{m,j}(n)\})$ is the conditional channel capacity given by [13]

$$C_j\left(\frac{E_s}{N_0},\{H_{m,j}(n)\}\right)=\frac{1}{N_c}\sum_{n=0}^{N_c-1}\log_2\left\{1+\gamma_j\left(\frac{E_s}{N_0},\{H_{m,j}(n)\}\right)\right\}. \tag{16}$$

A closed-form or convenient expression for numerical calculation has not been found for integral in Eq. (15) and thus, we resort to the numerical computation approach.

Using Eq. (14), we can rewrite Eq. (15) as

$$C_j\left(\frac{E_s}{N_0}\right)=\frac{1}{N_c}\int_0^{\infty}\cdots\int_0^{\infty}\sum_{n=0}^{N_c-1}\log_2\left\{1+\gamma_j\left[\frac{E_s}{N_0},\{H_{m,j}(n)\}\right]\right\}\times\wp[\{H_{m,j}(n)\}]\prod_n dH_{m,j}(n), \tag{17}$$

where $\gamma_j(E_s/N_0,\{H_{m,j}(n)\})$ at the jth terminal T_j is given by Eq. (14).

The evaluation of the average capacity is done by Monte-Carlo numerical computation method as follows. A set of path gains $\{h_{m,l};\ l=0\sim L-1\}$ is generated using Eq. (2) to obtain channel gains $\{H_{m,j}(n);\ n=0\sim N_c-1\}$. Then, the ergodic capacity given by Eq. (17) is computed by repeating the computation of the conditional channel capacity

$$2\sigma^2 = \begin{cases} \frac{2N_0}{T_c N_c}\frac{1}{N_c}\sum_{n=0}^{N_c-1}|w_{0,\bar{j}}(n)w_{1,j}(n)|^2 + \frac{\frac{P_s}{N_c}\sum_{n=0}^{N_c-1}|\hat{H}_{0,\bar{j}}(n)\hat{H}_{1,j}(n)|^2 - |\tilde{H}|^2}{|H_{0,\bar{j}}(n)|^2+|H_{1,j}(n)|^2+(\frac{E_s}{2N_0})^{-1}}, & \text{SC;} \\ \frac{2N_0}{T_c N_c}|w_{0,\bar{j}}(n)w_{1,j}(n)|^2, & \text{OFDM.} \end{cases} \quad (13)$$

$$\gamma_j\left[\frac{E_s}{N_0}, \{H_{m,j}(n)\}\right] = \begin{cases} \frac{|\tilde{H}|^2}{(\frac{E_s}{2N_0})^{-1}\frac{1}{N_c}\sum_{n=0}^{N_c-1}[|H_{0,\bar{j}}(n)|^2+|H_{1,j}(n)|^2+(\frac{E_s}{2N_0})^{-1}]|w_j(n)|^2+\frac{1}{N_c}\sum_{n=0}^{N_c-1}|\hat{H}|^2-|\tilde{H}|^2}, & \text{SC;} \\ \frac{E_s}{2N_0}\frac{|H_{0,\bar{j}}(n)|^2|H_{1,j}(n)|^2}{|H_{0,\bar{j}}(n)|^2+|H_{1,j}(n)|^2+(\frac{E_s}{2N_0})^{-1}}, & \text{OFDM.} \end{cases} \quad (14)$$

TABLE II
SIMULATION PARAMETERS.

Transmitter	Data modulation	QPSK
	Block size	N_c=256
	GI	$N_g=32$
Channel	L-path block Rayleigh fading with $\Delta=1$	
Receiver	FDE	MMSE, ZF
	Channel Estimation	Ideal

for the given set of channel gains $\{H_{m,j}(n)\}$ using Eq. (14) a sufficient number of times.

IV. NUMERICAL RESULTS AND DISCUSSIONS

Numerical evaluation parameters are summarized in Table II. We assume the system without knowledge of the channel state information at the transmitters, but the perfect channel state information at the receivers (i.e., relay and destinations). In our evaluation we assume $N_c = 256$, GI length of $N_g = 32$ samples and ideal coherent QPSK data modulation/demodulation with $E[|d_j(t)|^2] = 1$. The propagation channel is an $L = 16$-path block Rayleigh fading channel, where $\{h_{l,m,j};\ l = 0 \sim L-1\}$ are zero-mean independent complex variables with $E[|h_{l,m,j}|^2] = 1/L$. We assume that the maximum time delay of the channel is less than the GI length (i.e., $L < N_g$ with $\Delta = 1$) and that all paths in any channel are independent with each other. We assume no shadowing loss and pathloss.

Figure 2 illustrates the achievable ergodic capacity as a function of the average signal energy per bit-to-AWGN power spectrum density ratio E_b/N_0 $(=0.5\times(E_s/N_0)\times(1+N_g/N_c))$ with the number of paths $L = 16$ for both SC-FDE and OFDM.

Fig. 2(a) illustrate the ergodic capacity comparison between (i) analog network coding (i.e., ANC) and (ii) conventional point-to-point communication (i.e., w/o relay). The figure shows that the ergodic capacity of OFDM without relaying is slightly increased in comparison with ANC scheme. This is because in ANC a total available power is divided between the source terminals and the relay, while without relaying the total power is available at the source terminal. The figure shows that OFDM with ANC achieves a higher channel capacity in comparison with SC-FDE.

Fig. 2(b) illustrate the ergodic capacity comparison between (i) analog network coding (i.e., ANC) and (ii) cooperative relay networking (i.e., CoopNet) [16]. It can be seen from the figure that the performance with cooperative relay networking is slightly better than ANC due to the gain obtained through diversity combining. The figure shows that OFDM with ANC achieves a higher capacity in comparison with SC-FDE under the same channel conditions. Thus, an appropriate diversity scheme for ANC should be developed to further improve the capacity similar as cooperative relay networking case.

V. CONCLUSION

In this paper, we evaluated the ergodic capacity of bi-directional OFDM and SC-FDE transmission with ANC in a frequency-selective fading channel. The ergodinc capacity for SC-FDE is evaluated based on the Gaussian approximation of ISI after FDE. It was shown that OFDM with ANC achieves a higher capacity in comparison with SC-FDE under the same channel conditions. On the other hand, the ergodic capacity of cooperative relay networking scheme is slightly better in comparison with ANC case due to diversity combining gain in cooperative relaying.

Furthermore, an appropriate diversity scheme for ANC should be presented to further improve the capacity similar as cooperative relay networking case. This is left as interesting future work.

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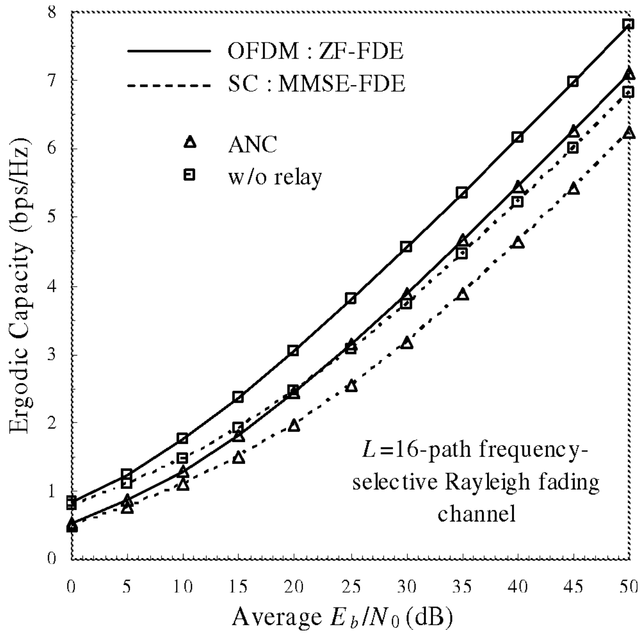

(a) ANC vs. point-to-point communication

(b) ANC vs. cooperative relaying

Fig. 2. Ergodic channel capacity.

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