Lattice Reduction-aided Uplink Multi-user MIMO in OFDM Cellular Systems

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Abstract— **In cellular systems, multi-user multi-input multi-output (MIMO) multiplexing can increase the number of simultaneous users per cell; however co-channel interference (CCI) from other cells limits the number of users/cell (defined as the cellular capacity). The well-known zero-forcing detection (ZFD) and minimum mean square error detection (MMSED) are computationally efficient detection methods, but their diversity order decreases as the number of users increases. The multi-user signal detection method which has the full diversity order and less complexity is required in order to enhance the cellular capacity. In this paper, we introduce the lattice reduction (LR) aided ZFD and MMSED to the uplink multi-user MIMO using orthogonal frequency division multiplexing (OFDM) and investigate the uplink cellular capacity. LR using Lenstra-Lenstra-Lovasz (LLL) algorithm is a promising technique to improve the performance of ZFD and MMSED. We show that LR aided ZFD and MMSED can obtain larger uplink capacity than conventional ZFD and MMSED at the cost of relatively low increase of computational complexity.**

Keywords-component; Multi-user MIMO, lattice reduction, uplink capacity

I. INTRODUCTION

In cellular systems, the same carrier frequency is reused at spatially separated different cells in order to efficiently utilize the limited bandwidth [1]. Multi-user multi-input multi-output (MIMO) multiplexing [2, 3] can increase the number of simultaneous users per cell without increasing the signal bandwidth; however, co-channel interference (CCI) from other cells limits the number of users/cell (defined as the cellular capacity). The uplink cellular capacity depends on the signal detection method since the superposition of different user's transmit signals and CCI from surrounding co-channel cells is received.

The well-known MIMO signal detection methods are maximum likelihood detection (MLD), zero-forcing detection (ZFD), and minimum mean square error detection (MMSED) [4]. MLD provides the full diversity order irrespective of the number of users, but has a prohibitively high computational complexity. On the other hand, ZFD and MMSED are a computationally efficient detection method, but their diversity order reduces as the number of users increases and hence, the achievable transmission performance degrades.

MIMO signal detection which can obtain the full diversity order with less computational complexity is indispensable to increase the cellular capacity from the practical implementation point of view. Lattice reduction (LR) using Lenstra-Lenstra-Lovasz (LLL) algorithm [5] is a promising technique to improve the performance of ZFD and MMSED [6]. The advantages of LR aided ZFD and MMSED are: they require only polynomial time algorithms [5] and always achieve the full diversity order irrespective of the number of users [7]. To the best of authors' knowledge, the LR aided uplink multi-user MIMO in cellular systems has not been fully studied. In this paper, we investigate the cellular capacity achievable by the LR aided ZFD and MMSED for the uplink multi-user MIMO in orthogonal frequency division multiplexing (OFDM) cellular systems.

The rest of the paper is organized as follows. Sect. II gives the CCI model. Sect. III presents the multi-user MIMO OFDM transmission system model. In Sect. IV, LR aided ZFD and MMSED are described. The simulation results and the uplink capacity achievable by the LR aided ZFD and MMSED are discussed in Sect. V. Sect. VI offers the conclusion.

II. CCI MODEL

Figure 1 illustrates the CCI model for the uplink multi-user MIMO in a cellular system. U users are simultaneously transmitting their data to the same BS using the same carrier frequency. It is assumed that the BS has N_r ($\geq U$) receive antennas while each user has a single transmit antenna.

In a cellular system, the whole channels are divided into a number of channel groups and each channel group is allocated to a different cell [1]. The number of different channel groups is defined as the cluster size N. When N is smaller, the number of channels per cell increases; but stronger CCI is received because the co-channel cells get closer. Therefore, there exists the optimum N that maximizes the uplink capacity. We consider 6 nearest co-channel cells since they are a dominant source of CCI which limits the cellular capacity. The cell of interest is indexed as c=0, and 6 nearest co-channel cells are indexed as c=1~6 (see Fig. 1).

In this paper, we measure the distribution of local average bit error rate (BER) by the Monte-Carlo simulation to find the outage probability, which is defined as the probability of the local average BER exceeding the required BER. We define the uplink capacity as the maximum number $U_{\max}$ of supportable users normalized by the cluster size N for the given allowable outage probability Q.

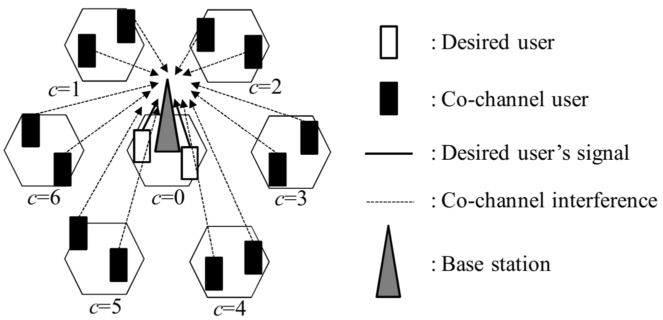


Fig. 1 CCI model of uplink multi-user MIMO in a cellular system.

III. MULTI-USER MIMO OFDM

Figure 2 illustrates the transmission system model of multi-user MIMO OFDM using N_c subcarriers. We consider the uplink transmission in the c=0th cell. At each user terminal transmitter, the binary information sequence is data-modulated and then, the data-modulated symbol sequence is divided into a sequence of blocks of N_c symbols each. The symbol block of u-th user in the c-th cell is expressed as $\{d_{u(c)}(k)\ ;\ k=0\sim N_c-1\}$. Then, N_c-point inverse fast Fourier transform (IFFT) is applied to generate the time-domain OFDM signal block as

$$s_{u(c)}(t)=\sqrt{2P}\sum_{k=0}^{N_c-1}d_{u(c)}(k)\exp\left(j2\pi t\frac{k}{N_c}\right), \tag{1}$$

where P is the transmit power and is the same for all users. The last N_g symbols in each block are copied and inserted as a cyclic prefix into the guard interval (GI) before transmission.

The transmitted OFDM signal block is assumed to go through a frequency-selective fading channel which is composed of L distinct paths. The channel impulse response between the u-th user in the c-th cell and the m-th receive antenna of the c=0th cell BS is given by

$$h_{m,u(c)}(t)=\sum_{l=0}^{L-1}h_{m,u(c)}^{(l)}(t)\cdot\delta\left(t-\tau_{u(c),l}\right), \tag{2}$$

with

$$h_{m,u(c)}^{(l)}(t)=\sqrt{r_{u(c)}^{-\alpha}\cdot 10^{-\eta_{u(c)}/10}}\cdot g_{m,u(c)}^{(l)}, \tag{3}$$

where $r_{u(c)}$, $\eta_{u(c)}$, and α denote the distance between the user and the c=0th cell BS, the shadowing loss, and the path-loss exponent, respectively, and $g_{m,u(c)}^{(l)}$ and $\tau_{u(c),l}$ are the complex-valued path gain and the time delay of the l-th path of the u-th user in the c-th cell, respectively, with $E\left[\sum_{l=0}^{L-1}\left|g_{m,u(c)}^{(l)}\right|^2\right]=1$.

At the c=0th cell BS, a superposition of U users' transmitted signals as well as CCI is received by N_r receive antennas. The GI-removed received signal block $\mathbf{y}(t)=\left[y_0(t)\ \cdots\ y_{N_r-1}(t)\right]^T$ can be expressed using the vector form as

$$\mathbf{y}(t)=\sum_{l=0}^{L-1}\mathbf{h}_0^{(l)}\mathbf{s}_0\left(\left(t-\tau_{0,l}\right)\bmod N_c\right)+\mathbf{i}(t)+\mathbf{n}(t), \tag{4}$$

where $\mathbf{i}(t)=\sum_{c=1}^{6}\sum_{l=0}^{L-1}\mathbf{h}_c^{(l)}\mathbf{s}_c\left(t-\tau_{c,l}\right)$ is the $N_r\times 1$ CCI vector and $\mathbf{n}(t)=\left[n_0(t)\ \cdots\ n_{N_r-1}(t)\right]^T$ is the $N_r\times 1$ noise vector. $\mathbf{h}_c^{(l)}$ is the $N_r\times U$ path gain matrix associated with the l-th path and $\mathbf{s}_c(t)=\left[s_{0(c)}(t)\ ,\cdots,\ s_{u(c)}(t),\cdots,s_{U-1(c)}(t)\right]^T$ is the $U\times 1$ transmitted signal vector. The $(m,u(c))$-th element of $\mathbf{h}_c^{(l)}$ is represented by $h_{m,u(c)}^{(l)}(t)$.

The received signal block $\mathbf{y}(t)$ is transformed by N_c-point fast Fourier transform (FFT) into the frequency-domain signal $\mathbf{Y}(k)=\left[Y_0(k)\ ,\cdots,\ Y_m(k),\cdots,Y_{N_r-1}(k)\right]^T$ as

$$\begin{aligned}\mathbf{Y}(k)&=\sum_{t=0}^{N_c-1}\mathbf{y}(t)\exp\left(-j\frac{2\pi k}{N_c}t\right)\\&=\mathbf{H}_0(k)\mathbf{S}_0(k)+\mathbf{I}(k)+\mathbf{N}(k),\ k=0\sim N_c-1\end{aligned}, \tag{5}$$

where $\mathbf{H}_0(k)$ is the $N_r\times U$ channel gain matrix, whose $(m,u(c))$-th element $H_{m,u(c)}(k)$ is given by

$$H_{m,u(c)}(k)=\sqrt{r_{u(c)}^{-\alpha}\cdot 10^{-\eta_{u(c)}/10}}\,G_{m,u(c)}(k), \tag{6}$$

where

$$G_{m,u(c)}(k)=\sum_{l=0}^{L-1}g_{m,u(c)}^{(l)}\exp\left(-j\frac{2\pi k}{N_c}\tau_{u(c),l}\right). \tag{7}$$

Finally, LR aided ZFD and MMSED are carried out using $\mathbf{Y}(k)$.

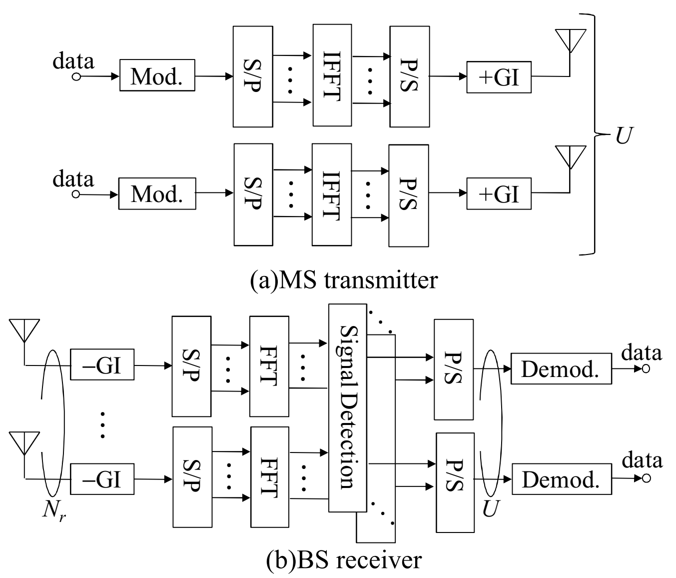


Fig. 2 Transmission system model of multi-user MIMO OFDM.

IV. LR Aided ZFD and MMSED

Without the LR, the diversity order of ZFD and MMSED is given by N_r–U+1 [9]. On the other hand, by the use of LR, the full diversity order of N_r can always be achieved [7].

(a) *LR-ZFD*

The purpose of introducing the LR is to transform $\mathbf{H}_0(k)$ into a new matrix $\tilde{\mathbf{H}}_0(k)$ consisting of near-orthogonal column vectors. The signal detection using $\tilde{\mathbf{H}}_0(k)$ produces less noise enhancement compared to that using $\mathbf{H}_0(k)$ [6]. In this paper, we realize the LR by using the LLL algorithm [5]. The detail of LLL algorithm is described in Appendix. By applying the LLL algorithm to $\mathbf{H}_0(k)$, we obtain $\tilde{\mathbf{H}}_0(k)=\mathbf{H}_0(k)\mathbf{T}(k)$, where $\mathbf{T}(k)$ is the transform matrix. Equation (5) can be rewritten as

$$\begin{aligned}\mathbf{Y}(k)&=\mathbf{H}_0(k)\mathbf{S}_0(k)+\mathbf{I}(k)+\mathbf{N}(k)\\&=\mathbf{H}_0(k)\mathbf{T}(k)\mathbf{T}^{-1}(k)\mathbf{S}_0(k)+\mathbf{I}(k)+\mathbf{N}(k),\\&=\tilde{\mathbf{H}}_0(k)\tilde{\mathbf{S}}_0(k)+\mathbf{I}(k)+\mathbf{N}(k)\end{aligned}\quad(8)$$

where $\tilde{\mathbf{S}}_0(k)=\mathbf{T}^{-1}(k)\mathbf{S}_0(k)$ and $\tilde{\mathbf{H}}_0(k)=\mathbf{H}_0(k)\mathbf{T}(k)$ are the transformed signal vector and equivalent channel matrix, respectively.

The soft decision output of LR-ZFD is expressed as

$$\tilde{\mathbf{S}}_{0,\mathrm{ZF}}(k)=\mathbf{W}(k)\mathbf{Y}(k),\quad(9)$$

where

$$\mathbf{W}(k)=\left(\tilde{\mathbf{H}}_0^H(k)\tilde{\mathbf{H}}_0(k)\right)^{-1}\tilde{\mathbf{H}}_0^H(k).\quad(10)$$

Hard decision on $\tilde{\mathbf{S}}_{0,\mathrm{ZF}}(k)$ is done first and then, $\mathbf{S}_{0,\mathrm{ZF}}(k)$ is obtained from $\mathbf{S}_{0,\mathrm{ZF}}(k)=\mathbf{T}(k)\tilde{\mathbf{S}}_{0,\mathrm{ZF}}(k)$.

(b) *LR-MMSED*

For LR-MMSED, the $(N_r+U)\times U$ channel gain matrix $\mathbf{H}_{\mathrm{ext}}(k)$ and the $(N_r+U)\times 1$ received signal vector $\mathbf{Y}_{\mathrm{ext}}(k)$ are introduced [6], [8]. They are given as

$$\mathbf{H}_{\mathrm{ext}}(k)=\begin{bmatrix}\mathbf{H}_0(k)\\ \sqrt{\dfrac{\sigma_I^2+\sigma_n^2}{P}}\mathbf{I}_U\end{bmatrix}\text{ and }\mathbf{Y}_{\mathrm{ext}}(k)=\begin{bmatrix}\mathbf{Y}_0(k)\\ \mathbf{0}_U\end{bmatrix},\quad(11)$$

where $\mathbf{I}_U$ is an $U\times U$ unit matrix and $\mathbf{0}_U$ is an $U\times 1$ vector whose elements are all 0. σ_I^2 and σ_n^2 denote the average received CCI power and the noise power, respectively. The soft decision output of LR-MMSED is expressed as

$$\tilde{\mathbf{S}}_{0,\mathrm{MMSE}}(k)=\mathbf{W}_{\mathrm{ext}}(k)\mathbf{Y}_{\mathrm{ext}}(k)\quad(12)$$

with

$$\mathbf{W}_{\mathrm{ext}}(k)=\left(\tilde{\mathbf{H}}_{\mathrm{ext}}^H(k)\tilde{\mathbf{H}}_{\mathrm{ext}}(k)\right)^{-1}\tilde{\mathbf{H}}_{\mathrm{ext}}^H(k),\quad(13)$$

where $\tilde{\mathbf{H}}_{\mathrm{ext}}(k)=\mathbf{H}_{\mathrm{ext}}(k)\mathbf{T}_{\mathrm{ext}}(k)$. By applying LLL algorithm to $\mathbf{H}_{\mathrm{ext}}(k)$, $\mathbf{T}_{\mathrm{ext}}(k)$ is obtained. Similar to LR-ZF, hard decision on $\tilde{\mathbf{S}}_{0,\mathrm{MMSE}}(k)$ is done first and then, $\mathbf{S}_{0,\mathrm{MMSE}}(k)$ is obtained from $\mathbf{S}_{0,\mathrm{MMSE}}(k)=\mathbf{T}(k)\tilde{\mathbf{S}}_{0,\mathrm{MMSE}}(k)$.

V. Computer Simulation

A. *Simulation procedure*

Table 1 shows the simulation condition. The channel is assumed to be a frequency-selective block Rayleigh fading having a symbol-spaced L-path uniform power delay profile (i.e., $E\left[\left|g_{m,u(c)}^{(l)}\right|^2\right]=1/L$). The transmit power P is set so that the average received bit energy-to-noise power spectrum density ratio E_b/N_0 from a user at the cell edge is equal to 10dB.

First, U users' locations are randomly generated in each cell for the given cluster size N. Next, the path-loss and the log-normally distributed shadowing loss are generated for each user. Then, an L-path block Rayleigh fading associated with each user is generated and signal transmission is simulated to measure the local average BERs of U users in the c=0th cell. This BER measurement is repeated a sufficient number of times by randomly changing the user locations to obtain the complementary cumulative distribution function (CCDF) of the local average BER. The outage probability is defined as the probability that the local average BER exceeds the required BER. If the outage probability is less than the allowable outage probability Q, the number U of users is incremented by one. We define the uplink capacity as the maximum number $U_{\max}$ of supportable users normalized by the cluster size N. In this paper, we set the required BER and the allowable outage probability to BER=10^{-3} and Q=0.1, respectively.

Table 1. Simulation condition

Transmitter	Data modulation	QPSK
	Number of users per cell	U=1~8
	Number of subcarriers	N_c=64
	GI length	N_g=16
Channel	Fading type	Frequency selective block Rayleigh
	Power delay profile	L=16-path uniform
	Path-loss exponent	α=3.5
	Standard deviation of shadowing loss	σ=7.0 [dB]
	Average received E_b/N_0 from cell edge	10 [dB]
Receiver	Number of receive antennas	N_r=8
	Channel estimation	Ideal
	LLL parameter	δ=0.75
Required quality	Required BER	10^{-3}
	Allowable outage probability	Q=0.1

B. Uplink capacity

Figure 3 illustrates the outage probability as a function of the number of users when N=21. Using conventional ZFD and MMSED, the outage probability significantly increases with U since their diversity order is given by N_r-U+1. On the other hand, the diversity order of LR-ZFD and LR-MMSED is always N_r and therefore, the outage probability is much lower than those of conventional ZFD and MMSED.

Figure 4 shows the uplink capacity, U_{max}/N, as a function of the cluster size N. The value of U_{max} is also indicated near each mark plotted in the figure. When N is small, the CCI power is too strong and therefore, the difference of U_{max} between MMSED and LR-MMSED (also between ZFD and LR-ZFD) is at most 1 user only. As N increases, the CCI power gets weaker and therefore, more users should be accommodated. However, even though N increases (i.e., the CCI power gets weaker), U_{max} is limited to 4 when either ZFD or MMSED is used (see Fig. 4). This is because the diversity order of conventional ZFD and MMSED is equal to N_r-U+1 and hence, increasing N doesn't lead to the performance improvement. By contrast, U_{max} increases for both LR-ZFD and LR-MMSED as N increases. We compare the maximum value of U_{max} in a range of N=1 to 25; both LR-ZFD and LR-MMSED achieve U_{max}=8 while ZFD and MMSED achieve U_{max}=4 only.

The conventional ZFD always provides the same uplink capacity as the conventional MMSED. The reason for this is discussed below. The conventional MMSED produces less noise enhancement than the conventional ZFD. However, when the number U of users is much smaller than the number N_r of receive antennas, the conventional ZFD produces very weak noise enhancement. Therefore, the conventional ZFD provides almost the same outage probability as MMSED in the case of $U \leq 4$ and consequently, the same uplink capacity. On the other hand, LR-ZFD and LR-MMSED provide slightly different uplink capacities. When N=21 and U=8 (see Fig. 3), the outage probability is lower than Q=0.1 for LR-MMSED while it is slightly higher for LR-ZFD. In this case, U=N_r and relatively large noise enhancement is produced in LR-ZFD. Consequently, when N=21, LR-MMSED can accommodate U_{max}=8 users while LR-ZFD can accommodate U_{max}=7 only.

Finally, we discuss the trade-off relationship between the maximum uplink capacity and the computational complexity for the conventional MMSED and LR-MMSED. In this paper, the computational complexity is measured as the number of complex multiply operations. The number of multiply operations is given by $N_c\{U^3+2(N_r+U)U^2+(N_r+U)U\}$ for the conventional MMSED and is shown in Table 2 for LR-MMSED. We measure the complexity of LLL algorithm by computer simulation since it depends on the channel condition. The maximum uplink capacity of conventional MMSED is 0.250 (this is achieved in the case of U=3, N=12), and that of LR-MMSED is 0.385 (in the case of U=5, N=13). Hence, the computational complexity which is required to achieve the maximum uplink capacity is about 1.7×10^4 for the conventional MMSED and is about 1.2×10^5 for LR-MMSED. Consequently, LR-MMSED can achieve 1.54 times higher capacity at the cost of about 7.3 times increased complexity. LR-MMSED can provide larger uplink capacity with relatively low increase in the computational complexity. This is because conventional MMSED is a computationally efficient detection method.

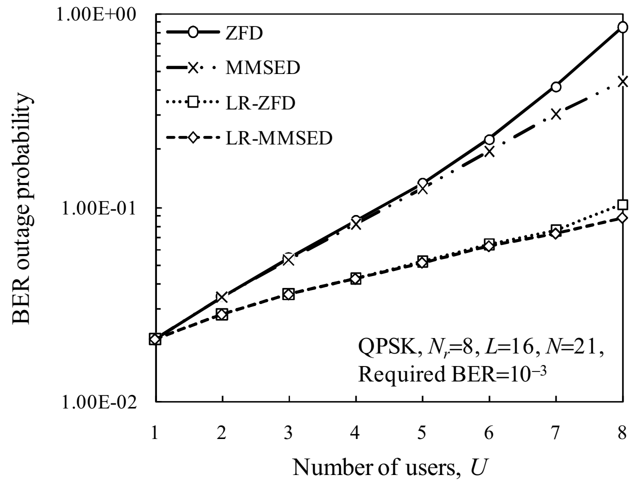
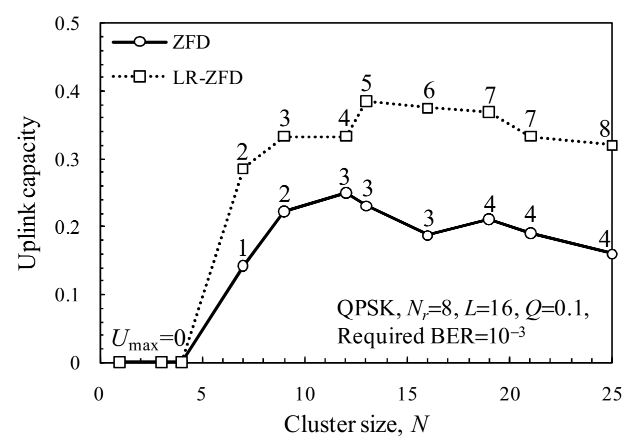
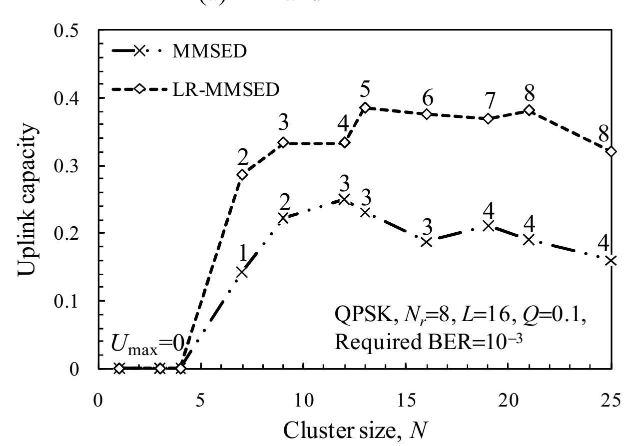


Fig. 3 Outage probability comparison.



(a)ZFD and LR-ZFD



(b)MMSED and LR-MMSED

Fig. 4 Uplink capacity comparison.

Table 2. Number of complex multiply operations of LR-MMSED

	No. of complex multiply operations
QR decomposition	$N_c(N_r+U)U^2$
Calculation of $\mathbf{T}^{-1}$	N_cU^3
Detection of $\tilde{\mathbf{s}}_0$	$N_c\{U^3+2(N_r+U)U^2+(N_r+U)U\}$
Detection of $\mathbf{s}_0$	N_cU^2

VI. CONCLUSIONS

In this paper, we introduced the lattice reduction (LR) aided ZFD and MMSED to the uplink multi-user MIMO OFDM and investigated the uplink cellular capacity. LR-ZFD and LR-MMSED can provide much lower outage probability than ZFD and MMSED. It was shown that when N_r=8, ZFD and MMSED cannot accommodate more than 4 users even if the CCI power is weak. However, LR-ZFD and LR-MMSED can always achieve N_r diversity order irrespective of the number of users and therefore, larger uplink capacity can be obtained. We showed that LR-MMSED achieves 1.54 times higher capacity than conventional MMSED at the cost of 7.3 times increased computational complexity.

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APPENDIX

The lattice of matrix $\mathbf{H}_0=\lfloor\mathbf{H}_{0,0} \ \cdots \ \mathbf{H}_{0,U-1}\rfloor$ is defined as

$$\mathrm{L}(\mathbf{H}_0)=\mathrm{L}(\mathbf{H}_{0,0},\ldots,\mathbf{H}_{0,U-1})\equiv\sum_{k=0}^{U-1}x\mathbf{H}_k \ , \quad x\in\mathbf{Z}\,, \qquad (A.1)$$

where $\mathbf{H}_{0,0},\ldots,\mathbf{H}_{0,U-1}$ are column vectors of matrix $\mathbf{H}_0$ and called the lattice basis. $\mathbf{Z}$ represents the infinite integer space.

The LLL algorithm [5] is one of the methods to perform the lattice reduction. Firstly, QR decomposition is applied to obtain $\mathbf{H}_0$=$\mathbf{QR}$, where $\mathbf{Q}$ is an $N_r\times U$ matrix which satisfies $\mathbf{Q}^H\mathbf{Q}$=$\mathbf{I}_U$ and $\mathbf{R}$ is a $U\times U$ upper triangular matrix. Let $\mathbf{Q}_k$ be the k-th column vector of $\mathbf{Q}$ and $R_{l,k}$ be the (l, k)-th element of $\mathbf{R}$. Then, $\mathbf{Q}$ and $\mathbf{R}$ are obtained from [10]

$$R_{l,k}=\begin{cases}\mathbf{Q}_l^H\mathbf{H}_k & \text{if } l\neq k\\ \left\|\mathbf{H}_k-\sum_{l=0}^{k-1}R_{l,k}\mathbf{Q}_l\right\| & \text{if } l=k\end{cases}, \qquad (A.2)$$

$$\mathbf{Q}_k=\frac{1}{R_{k,k}}\left(\mathbf{H}_k-\sum_{l=0}^{k-1}R_{l,k}\mathbf{Q}_l\right). \qquad (A.3)$$

The inputs to the LLL algorithm are $\mathbf{Q}$ and $\mathbf{R}$ and its outputs are $\tilde{\mathbf{Q}}$ of size $N_r\times U$, $\tilde{\mathbf{R}}$ of size $U\times U$, and $\mathbf{T}$ of size $U\times U$. The elements of $\tilde{\mathbf{R}}$ satisfy the following two conditions:

$$\left|\tilde{R}_{l,k}\right|\leq\frac{1}{2}\left|\tilde{R}_{l,l}\right| \qquad (0\leq l<k\leq U-1), \qquad (A.4)$$

$$\delta\left|\tilde{R}_{k-1,k-1}\right|^2\leq\left|\tilde{R}_{k,k}\right|^2+\left|\tilde{R}_{k-1,k}\right|^2 \quad (k=1,\ldots,U-1)\,. \qquad (A.5)$$

The range of δ is $1/4<\delta\leq1$ [5]. Since $\delta=3/4$ is often used [5, 6, 11], $\delta=3/4$ is also used in Sec. V.

$\mathbf{H}_0$ and its lattice reduced matrix $\tilde{\mathbf{H}}_0$ are related as [6]

$$\tilde{\mathbf{H}}_0=\mathbf{H}_0\mathbf{T}\,, \qquad (A.6)$$

where $\mathbf{T}$ is a unimodular matrix [6], which consists of only Gaussian integer and its determinant is 1 or −1.