Sparse Least Mean Fourth Filter with Zero-Attracting ℓ_1-Norm Constraint

Guan Gui
Dept. of Communications Engineering
Graduate School of Engineering
Tohoku University, Sendai, Japan
Email: gui@mobile.ecei.tohoku.ac.jp

Fumiyuki Adachi
Dept. of Communications Engineering
Graduate School of Engineering
Tohoku University, Sendai, Japan
Email: adachi@ ecei.tohoku.ac.jp

***Abstract*—Traditional stable adaptive filter was used normalized least-mean square (NLMS) algorithm. However, identification performance of the traditional filter was especially vulnerable to degradation in low signal-noise-ratio (SRN) regime. Recently, adaptive filter using normalized least-mean fourth (NLMF) is attracting attention in adaptive system identifications (ASI) due to its high identification performance and stability. In the case of sparse system, however, the NLMF filter cannot identify effectively due to the fact that its algorithm neglects the inherent sparse structure. In this paper, we proposed a sparse NLMF filter using zero-attracting ℓ_1-norm constraint to exploit the sparsity and to improve the identification performance. Effectiveness of the proposed filter is confirmed from two aspects: 1) stability is derived equivalent to well-known stable NLMS filter; 2) identification performance of the proposed is verified by mean square deviation (MSD) standard in computer simulations. When comparing with conventional adaptive filter, the proposed one can achieve much better identification performance especially in low SNR regime.**

Keywords*—normalized least-mean square (NLMS), normalized least-mean fourth (NLMF), zero-attracting ℓ_1-norm constraint normalized least-mean fourth (ZAC-NLMF), adaptive filter, sparse system identification.

I. Introduction

A. Background and motivation

Adaptive filters are often applied in many adaptive system identifications (ASI), as shown in Fig. 1, such as noise canceling, channel equalization, radar target localization and etc. The typical stable filter uses normalized least-mean fourth (NLMF) algorithm since NLMS filter depends solely on the input signal power [1] for a given step size of gradient descend. It is well known that least mean fourth (LMF) filter [1] outperforms the NLMS filter in achieving a better balance between convergence and steady-state performances. Unfortunately, standard LMF filter is unstable due to the fact that its stability depends on three factors: input signal power, noise power as well as filter weight initialization [2]. In other words, any variation of the three factors may decrease the stability of LMF filter. Hence, it is necessary to develop stable LMF filter for ASI. Recently, a stable LMF filter was only proposed in low signal-to-noise ratio (SNR) regime [3].

To take the advantage of LMF as well as NLMS, stable normalized LMF (NLMF) filter has been proposed in [4], [5]. Recently, many systems were confirmed exhibiting sparse structure as shown in Fig. 2. In other words, most of the filter coefficients are modeled as zeros. However, the proposed NLMF filter [4], [5] always neglects the inherent sparse structure information and therefore it may degrade the identification performance.

To exploit the sparsity, sparse filter using zero-attracting ℓ_1-norm constraint LMS (ZAC-LMS) algorithm has been proposed in [6]. Motivated by this method, an improved sparse filter using zero-attracting ℓ_1-norm constraint NLMS (ZAC-NLMS) was proposed in [7], [8].

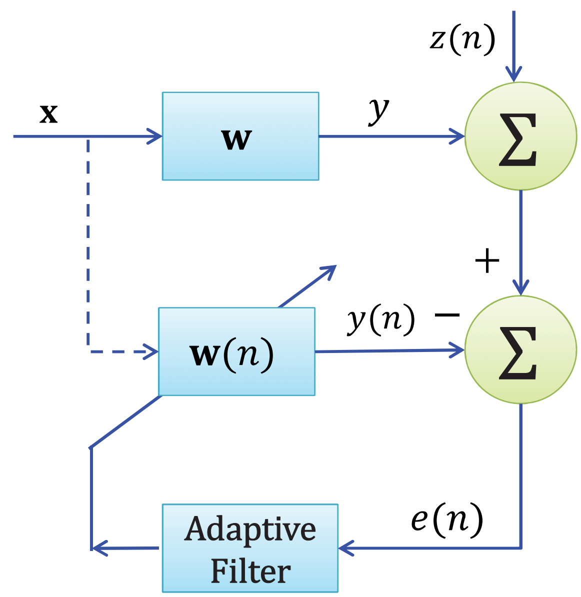

Fig. 1. System identification using adaptive filter

B. Main contribution

Unlike the two proposed filters in [6–8], a stable sparse filter using zero-attracting ℓ_1-norm constraint NLMF (ZAC-NLMF) is proposed in this paper. The proposed filter provides a better identification performance than ZAC-NLMS one. It is effective for exploiting system sparsity and stable for any conditions of input signal, noise, as well as initial setting of the filter. For one thing, the updating normalization stabilizes the filter against increasing input power and the infinity of the input distribution. For another, the estimation error normalization term stabilizes the filter against increasing noise power and increasing initial weight deviation. At last, the zero-attracting ℓ_1 -norm constraint sparse penalty replaces small filter coefficients with zero so that the sparsity can be exploited. Note that when the values of step-size is within the range (0,2), the stability of the ZAC-NLMF filter is similar to that of the ZAC-NLMS one [4], [5]. Performance of the proposed filter is evaluated by the computer simulations via mean square deviation (MSD) standard.

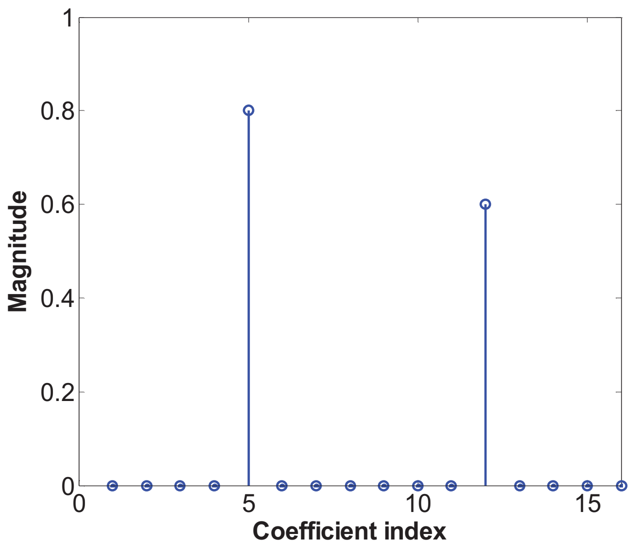

Fig. 2 . An example of sparse system where model length is 16 and number of nonzero coefficient is 2.

C. Relattions to past research

In our previous work [3], an stable sparse LMF filter, using fourth-order power optimization criterion was proposed to improve the system identification performance in low SNR regime, e.g., $SNR \leq 5dB$. Different from the previous method, the proposed filter in this paper is more flexible and it can maintain its stability in different SNR regimes. In [9], we proposed an reweighted zero-attracting ℓ_1 -norm constraint NLMF (RZAC-NLMF) algorithm for exploiting sparsity in sparse channel. In addition, different reweighted factors were evaluated for RZAC-NLMF. Even though ZAC-NLMF algorithm was also mentioned in this paper, only sparse channel estimation was considered. In this paper, sparse ZAC-NLMF filter is considered for ASI. In [10], least mean square/fourth (LMS/F) algorithm was also proposed for ASI. To exploit the system sparsity, sparse LMS/F algorithm was also proposed in [11]. One can find that the proposed method is based on standard LMS/F algorithm [10]. Unlike the proposed method in [11], sparse ZAC-NLMF filter is based on the standard LMF algorithm [1].

The remainder of rest paper is organized as follows. A standard NLMF filter is described in Section II. In section III, sparse NLMF filter using zero-attracting ℓ_1-norm constraint is proposed. Computer simulation results are given in Section IV in order to evaluate and compare performances of the proposed NLMF filter. Finally, we conclude the paper in Section V.

II. Standard NLMF Filter

Assume that signal $\mathbf{x}(n)$ is input to the system with unknown K -sparse filter FIR coefficients vector $\mathbf{w} = [w_{0,} w_1, \dots, w_{N-1}]^{\mathrm{T}}$, then its observed output signal y(n) is given by

$$y(n) = \mathbf{w}^T \mathbf{x}(n) + z(n), \quad (1)$$

where $\mathbf{x}(n) = [x(n), x(n-1), \dots, x(n-N+1)]^T$ denotes the vector of input signal x(n), and z(n) is the observation noise which is assumed to be independent from $\mathbf{x}(n)$. The objective is to adaptively identify the unknown FIR coefficients vector $\mathbf{w}$ using the input signal $\mathbf{x}(n)$ and the observed output $y(n)$. Conventionally, cost function of LMF filter is given by

$$L_1(n) = \frac{1}{4} e^4(n), \quad (2)$$

where $e(n) = y(n) - \mathbf{w}^T(n)\mathbf{x}(n)$ denotes filter mismatching error. The update equation of the filter can be written as

$$\begin{aligned} \mathbf{w}(n+1) &= \mathbf{w}(n) + \mu_f \frac{\partial L_1(n)}{\partial \mathbf{w}(n)} \\ &= \mathbf{w}(n) + \mu_f e^3(n)\mathbf{x}(n), \end{aligned} \quad (3)$$

where $\mu_f \in (0,2)$ denotes the step-size of gradient descend. Unfortunately, the above filter is not stable [2] and hence it cannot be applied in adaptive system identification. Thanks to the contribution in [6,7], a stable NLMF filter was proposed and its update equation is given as

$$\begin{aligned} \mathbf{w}(n+1) &= \mathbf{w}(n) + \mu_f \frac{e^3(n)\mathbf{x}(n)}{\|\mathbf{x}(n)\|_2^2 \left(\|\mathbf{x}(n)\|_2^2 + e^2(n)\right)} \\ &= \mathbf{w}(n) + \mu_f(n) \frac{e(n)x(n)}{\|\mathbf{x}(n)\|_2^2}, \end{aligned} \quad (4)$$

where $\mu_f(n) = \mu e^2(n) / \left(\|x(n)\|_2^2 + e^2(n)\right)$ denotes a variable step-size and $\|\cdot\|_2$ is the Euclidean norm operator and $\|\mathbf{x}\|_2^2 = \sum_{i=0}^{N-1} |x_i|^2$. Here, it is easy to observe that

$$\begin{cases} \mu_f(n) \to \mu_f, & when \;\; e^2(n) \gg \|\mathbf{x}(n)\|_2^2 \\ \mu_f(n) \to \dfrac{\mu_f}{2}, & when \;\; e^2(n) \gg \|\mathbf{x}(n)\|_2^2 \\ \mu_f(n) \to 0, & when \;\; e^2(n) \ll \|\mathbf{x}(n)\|_2^2 \end{cases} \quad (5)$$

Since the step-size $\mu_f(n)$ is variant, one can find that, instantaneous estimation error $\mathrm{e}^2(n)$ much larger than the $\|\mathbf{x}(n)\|_2^2$, larger $\mu_f(n)$ is utilized for reducing computational complexity; Conversely, instantaneous estimation error $\mathrm{e}^2(n)$ decreases, step-size $\mu_f(n)$ is also reducing for ensuring filter's stability. Therefore, the stability of NLMF approaches to NLMS.

III. PROPOSED SPARSE FILTER

To take the advantage of the sparsity of sparse system, sparse LMS filter [6] and improved sparse NLMS filter [6,7] have been proposed, respectively. Different from the previous methods, cost function of sparse LMF filter is given by

$$L_2(n) = \frac{1}{4}e^4(n) + \lambda_f \|w(n)\|_1, \tag{6}$$

where $\lambda_f > 0$ is a regularization parameter which trades off the fourth-order mismatching error and sparseness of LMF filter; $\|\cdot\|_1$ denotes zero-attracting ℓ_1-norm constraint function. The updating equation of sparse LMF filter can be written as

$$\begin{aligned}\mathbf{w}(n+1) &= \mathbf{w}(n) + \mu_f \frac{\partial L_2(n)}{\partial \mathbf{w}(n)} \\ &= \mathbf{w}(n) + \mu_f e^3(n)\mathbf{x}(n) - \beta \mathrm{sign}\big(\mathbf{w}(n)\big),\end{aligned} \tag{7}$$

where $\beta = \mu_f \lambda_f$ and $\mathrm{sign}(\cdot)$ is the sign function which operates on every coefficients of filter independently and

$$\begin{cases} \mathrm{sign}(w) = 0, & when\ \ w = 0 \\ \mathrm{sign}(w) = 1, & when\ \ w > 0 \\ \mathrm{sign}(w) = -1, & when\ \ w < 0 \end{cases} \tag{8}$$

However, the stability of (7) still depends on input signal power, noise power and weight initialization of sparse NLMF. To improve its stability, based on the standard NLMF filter in (4), the update equation in Eq. (7) is modified as

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu_f(n)\frac{e(n)\mathbf{x}(n)}{\|\mathbf{x}(n)\|_2^2} - \beta \mathrm{sign}\big(\mathbf{w}(n)\big). \tag{9}$$

where $\mu_f(n) = \mu e^2(n)/\big(\|x(n)\|_2^2 + e^2(n)\big)$. According to above derivation of sparse NLMF filter, the stable filter algorithm can be written as Tab. I.

TAB. 1. ALGORITHM OF SPARSE NLMF FILTER

Input: Reference signal $\mathbf{x}$, output signal y, regularization parameter: β, step-size μ_f, updating error threshold δ, the maximum updating times n_{max}.

Output: modeled filter coefficients: $\mathbf{w}(n)$

Initialization: $n = 0$; updating error $e(0) = y$, filter coefficients vector: $\mathbf{w}(n) = 0$;

While $n \le n_{max}$ or $\|\mathbf{w}(n+1) - w\|_2^2 \ge \delta$

Run

Error updating: $e(n) = y(n) - \mathbf{w}^T(n)\mathbf{x}(n)$

Coefficients updating:

$$\mathbf{w}(n+1) = \mathbf{w}(n-1) + \mu_f(n)\frac{e(n)\mathbf{x}(n)}{\|\mathbf{x}(n)\|_2^2} - \beta \mathrm{sign}\big(\mathbf{w}(n)\big)$$

End

IV. COMPUTER SIMULATION

In this section, computer simulation adopts 1000 independent Monte-Carlo runs for averaging. Four filters, i.e., NLMS, NLMF, ZAC-NLMS and ZAC-NLMF will be evaluated by MSD standard which is defined as

$$\mathrm{MSD}\{\mathbf{w}(n)\} = \mathrm{E}\{\|\mathbf{w} - \mathbf{w}(n)\|_2^2\}, \tag{10}$$

where $\mathrm{E}\{\cdot\}$ denotes expectation operator, $\mathbf{w}$ and $\mathbf{w}(n)$ denote actual filter coefficient vector and its estimator, respectively. The FIR filter length is set as $N = 16$ and its number of nonzero coefficients is set as $K = 1$ and 4 respectively. The values of the nonzero FIR coefficients follow Gaussian distribution and the positions of coefficients are randomly allocated within the FIR filter length and $\|\mathbf{w}\|_2^2 = 1$. The received signal-to-noise ratio (SNR) is defined as $\mathrm{SNR} = 10\log(E_0/\sigma_n^2)$, where $E_0 = 1$ is normalized transmitted power and the noise power is given by $\sigma_n^2 = 10^{-SNR/10}$. These filters use the same step-size each time and two step-sizes. As for a typical example, $\mu_s = 0.2$ and $\mu_f = 1.2$ are considered. Likewise, the regularization parameter is set as $\lambda_s = 5 \times 10^{-3}\sigma_n^2$ and $\lambda_f = 5 \times 10^{-5}\sigma_n^2$ respectively.

In the first experiment, considering two sparse channels, i.e., $K = 1$ and 4, in the case of $SNR = 5dB$, we demonstrate that the convergence speed and steady-state performance of the four filters in Figs. 3 and 4, respectively. It can be observed that the proposed sparse NLMF filter is stable and it can obtain much better performance than sparse NLMS filter. Please note that standard NLMF filter can also achieve better performance than NLMS. In addition, as shown in Figs. 3 and 4, there are two performance gaps in between NLMS and ZAC-NLMS as well as between NLMF and ZAC-NLMF. One can deduce that performance curves of two sparse filters (i.e., ZAC-NLMS and ZAC-NLMF) have a relationship with the sparseness of filter coefficients' vector, i.e., number of nonzero coefficients, K. That is to say, by using either ZAC-NLMS or ZAC-NLMF, sparser filter may result in a better performance and vice versa. Unlike the two sparse filters, both NLMS and NLMF cannot exploit filter sparsity due to the fact that their performance curves are almost invariant even if number of nonzero coefficients variant. Hence, the proposed filter can exploit sparsity as for prior information and then it can improve system identification performance.

In the first experiment, considering two sparse channels, i.e., $K = 1$ and 4, in the case of $SNR = 5dB$, we demonstrate that the convergence speed and steady-state performance of the four filters in Figs. 3 and 4, respectively. It can be observed that the proposed sparse NLMF filter is stable and it can obtain much better performance than sparse NLMS filter. Please note that standard NLMF filter can also achieve better performance than NLMS. In addition, as shown in Figs. 3 and 4, there are two performance gaps in between NLMS and ZAC-NLMS as well as between NLMF and ZAC-NLMF. One can deduce that performance curves of two sparse filters (i.e., ZAC-NLMS and ZAC-NLMF) have a relationship with the sparseness of filter coefficients' vector, i.e., number of nonzero coefficients, K. That is to say, by using either ZAC-NLMS or ZAC-NLMF, sparser filter may result in a better performance and vice versa. Unlike the two sparse filters, both NLMS and NLMF cannot

exploit filter sparsity due to the fact that their performance curves are almost invariant even if number of nonzero coefficients variant. Hence, the proposed filter can exploit sparsity as for prior information and then it can improve system identification performance.

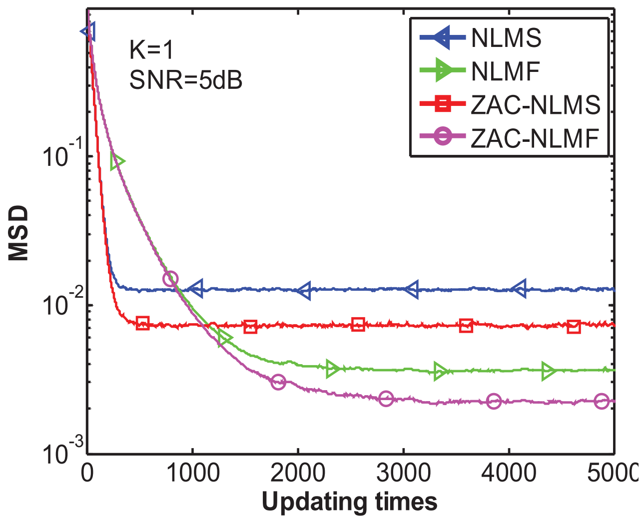

Fig. 3. MSD performance evaluation at $K = 1$ and $SNR = 5dB$.

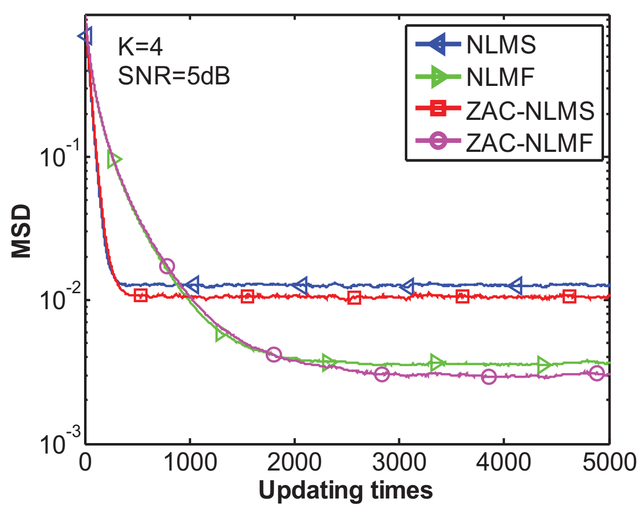

Fig. 4. MSD performance evaluation at $K = 4$ and $SNR = 5dB$.

V. Conclusion

Adaptive system identification requires that filter not only works reliably but also exploits sparse structure information efficiently. Based on the standard NLMF filter, in this paper, we proposed a novel ZAC-NLMF filter to exploit the system sparsity. The proposed filter was inspirited from the fact that LMF algorithm achieves better MSD performance than LMS in ASI. Thereby, LMF suppresses noise ability stronger than LMS due to LMF utilizes higher-order mean error statistics than LMS. To maintain the sparse LMF filter stable, NLMF algorithm was proposed for the proposed filter. Computer simulation results have shown the superior performance of the proposed filter compared with the existing filters.

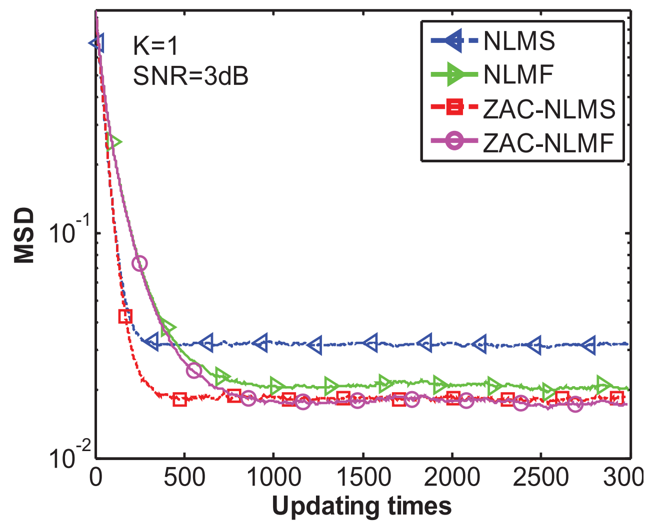

Fig. 5. MSD performance evaluation at $K = 1$ and $SNR = 3dB$.

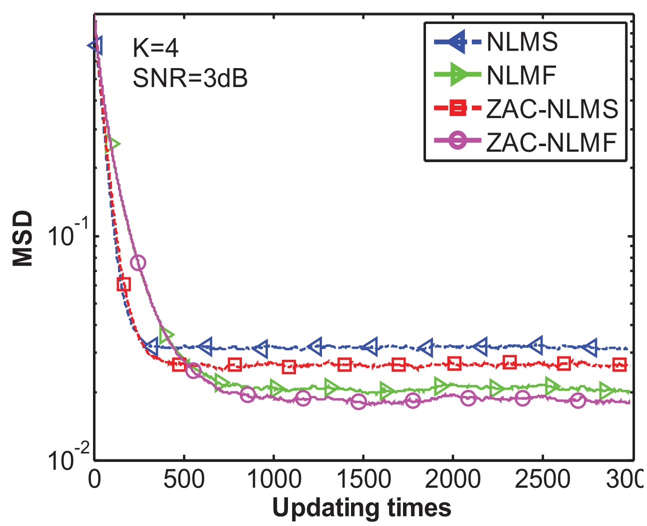

Fig. 6. MSD performance evaluation at $K = 4$ and $SNR = 3dB$

Acknowledgments

The authors would like to think Dr. Koichi Adachi of Institute for Infocomm Research for his valuable comments and suggestions. This work was supported in part by the Japan Society for the Promotion of Science (JSPS) postdoctoral fellowship and the National Natural Science Foundation of China under Grant 61261048.

Reference

[1] E. Walach and B. Widrow, "The least mean fourth (LMF) adaptive algorithm and its Family," *IEEE Transactions on Information Theory*, vol. 30, no. 2, pp. 275–283, 1984.

[2] B. Widrow and D. Stearns, *Adaptive signal processing*, no. 4. New Jersey: Prentice Hall, 1985.

[3] G. Gui and F. Adachi, "Sparse LMF algorithm for adaptive channel estimation in low SNR region," *International Journal of Communication Systems* (Wiely), 2013. DOl:10.1002/dac.2531

[4] E. Eweda, "Global stabilization of the least mean fourth algorithm," *IEEE Transactions on Signal Processing*, vol. 60, no. 3, pp. 1473–1477, Mar. 2012.

[5] E. Eweda and N. J. Bershad, "Stochastic analysis of a stable normalized least mean fourth algorithm for adaptive noise canceling with a white Gaussian reference," *IEEE Transactions on Signal Processing*, vol. 60, no. 12, pp. 6235–6244, 2012.

[6] Y. Chen, Y. Gu, and A. O. Hero III, "Sparse LMS for system identification," *in IEEE International Conference on Acoustics, Speech and Signal Processing* (ICASSP), April 19-24, 2009 - Taipei, Taiwan, no. 3, pp. 3125–3128.

[7] G. Gui, W. Peng, and F. Adachi, "Improved adaptive sparse channel estimation based on the least mean square algorithm," in *IEEE Wireless Communications and Networking Conference* (WCNC), Shanghai, China, April 7-10, 2013, pp. 3130–3134.

[8] G. Gui and F. Adachi, "Improved adaptive sparse channel estimation using least mean square algorithm," *EURASIP Journal on Wireless Communications and Networking,* vol. 2013, no. 1, pp. 1-18, 2013.

[9] G. Gui, A. Mehbodniya, and F. Adachi, "Adaptive sparse channel estimation using re-weighted zero-attracting normalized least mean fourth," in *2nd IEEE/CIC International Conference on Communications in China* (ICCC), Aug.12-14, 2013, Xian, China, 2013.

[10] G. Gui, W. Peng, and F. Adachi, "Adaptive system identification using robust LMS/F algorithm," *International Journal of Communication Systems* (Wiley), 2013. DOl:10.1002/dac.2517

[11] G. Gui, A. Mehbodniya, and F. Adachi, "Least mean square/fourth algorithm for adaptive sparse channel estimation," in *IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC), Sept. 8-11, 2013, London, UK*, 2013, pp. 1–5.